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CONTACT PROBLEM FOR A LAYER WITH TWO STAMPS

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A problem of impressing coaxial stamps of circular cross section into the upper and lower surface of a homogeneous elastic layer is studied. The bases of the stamps have axial symmetry. The parts of the layer surfaces lying oustide the contact zone are stress-free, there is no friction or coupling between the layer and the stamps. A system of two integral equations with two unknown functions is obtained, and provides a solution of the problem. The method of separating the singularities provides the way of reducing this system to the Fredholm equations of second kind. An approximate solution of the equations is obtained for the case of flat stamps under the assumptions that the two parameters entering the system are sufficiently small.

Problems of a layer with various boundary conditions were formulated and solved in many papers and books, e.g. [1, 2]. However, to the best of the author's knowledge, in all these problems the conditions at the boundary were assumed different only on one side of the layer; in the present problem the boundary conditions are mixed at both sides of the layer, and this results in a system of two integral equations.

1. Let λ and μ be the Lamè constants of the elastic layer

$$-d \leqslant z \leqslant d, \quad 0 \leqslant r < \infty \quad (r^2 = x^2 + y^2)$$

where u_r and u_z are the deformation components and σ_r, σ_z and τ_{rz} the stress components in the cylindrical coordinate system (see Fig. 1). The boundary conditions are

$$u_z |_{z=\pm d} = f_\pm (r), \quad r \leqslant r_\pm; \quad \sigma_z |_{z=\pm d} = 0, \quad r > r_\pm \tag{1.1}$$
$$\tau_{rz} |_{z=\pm d} = 0 \quad (f_\pm (r) = \pm g_\pm (r) \mp h_\pm)$$

Here $g_+ (r)$ and $g_- (r)$ are functions defining the bases of the stamps in terms of their radii r_+ and r_- , and h_+ , h_- are constant, so far unspecified, determining the depth of penetration of the stamps into the layer.

In the case of axial symmetry the stress and deformation components can be expressed in terms of a single biharmonic function. We choose this function in the form of a Hankel transform

$$\Phi (r, z) = \int_0^\infty G (\gamma, z) J_0 (\gamma r) \gamma \, d\gamma \tag{1.2}$$

$$G (\gamma, z) = (A + Bz) \operatorname{ch} (\gamma, z) + (C + Dz) \operatorname{sh} (\gamma z)$$

where $A = A(\gamma), \ldots, D = D(\gamma)$ are functions which should be found from the boundary conditions (1.1).

Substituting (1.2) into the known expressions for u_z, σ_z and τ_{rz} we obtain, with help of Φ,

$$u_z = \int_0^\infty \left(\frac{\partial^2 G}{\partial z^2} - \frac{\lambda + 2\mu}{\mu}\gamma^2 G\right)\gamma J_0(\gamma r)\,d\gamma \tag{1.3}$$

$$\sigma_z = \int_0^\infty \left[(\lambda + 2\mu)\frac{\partial^3 G}{\partial z^3} - (3\lambda + 4\mu)\frac{\partial G}{\partial z}\right]\gamma J_0(\gamma r)\,d\gamma$$

$$\tau_{rz} = \int_0^\infty \left[\lambda\frac{\partial^2 G}{\partial z^2} + (\lambda + 2\mu)\gamma^2 G\right]\gamma^2 J_1(\gamma r)\,d\gamma$$

Let us introduce the following notation:

$$-\sigma_z\big|_{z=\pm d} = p_\pm(r), \quad \bar{p}_\pm(\gamma) = \int_0^\infty p_\pm(r)\,r J_0(\gamma r)\,dr \tag{1.4}$$

Applying the Hankel transform inversion formula to the second equation of (1.3) and putting, consecutively, $z = d$ and $z = -d$, we obtain two equations for the unknown A, B, C and D. Another two equations yield the boundary conditions $\tau_{rz}|_{z=\pm d} = 0$.

Thus we can write the unknown $A, \ldots, D$ in terms of the new unknowns $\bar{p}_\pm(\gamma)$, and the function $G(\gamma, z)$ assumes the form

$$G(\gamma, z) = g_+(\gamma, z)\bar{p}_+(\gamma) + g_-(\gamma, z)\bar{p}_-(\gamma)$$

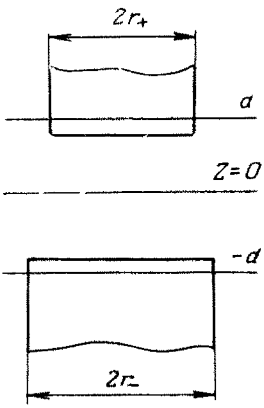

Fig. 1

Substituting now the above expression into the first equation of (1.3) and assuming, one after the other, $z = d$ and $z = -d$, we arrive at a system of two equations with the unknowns $\bar{p}_\pm(\gamma)$. Omitting the cumbersome, though elementary, manipulations, we arrive at the following results:

$$\int_0^\infty [K_\pm(\gamma)\bar{p}_+(\gamma) - K_\mp(\gamma)\bar{p}_-(\gamma)]\,\gamma J_1(\gamma r)\,d\gamma = \frac{E}{2(1-\sigma^2)} f'_\pm(r) \tag{1.5}$$

$$2K_\pm(\gamma) = \frac{\operatorname{ch} 2u + 1}{\operatorname{sh} 2u - 2u} \pm \frac{\operatorname{ch} 2u - 1}{\operatorname{sh} 2u + 2u}$$

$$u = \gamma d, \quad E = \mu\frac{3\lambda + 2\mu}{\lambda + \mu}, \quad \sigma = \frac{\lambda}{2(\lambda + \mu)}$$

The functions $K_\pm(\gamma)$ behave differently as $\gamma \to +\infty$. If $K_-(\gamma)$ tends to zero as the infinitesimal $4\gamma d \exp(-2\gamma d)$, then $K_+(\gamma) = 1 + O((4\gamma d \exp(-2\gamma d))^2)$. The latter causes the appearance of singularities in the left hand sides of the system (1.5).

Let us set $K_+(\gamma) = 1 + R_+(\gamma)$, $K_-(\gamma) = R_-(\gamma)$ and separate these singularities in the explicit form

$$\pm\int_0^\infty \bar{p}_\pm(\gamma)\,\gamma J_1(\gamma r)\,d\gamma + I_\pm(r) = \frac{E}{2(1-\sigma^2)} f_\pm'(r) \qquad (1.6)$$

$$I_\pm(r) = \int_0^\infty [R_\pm(\gamma)\,\bar{p}_+(\gamma) - R_\mp(\gamma)\,\bar{p}_-(\gamma)]\,\gamma J_1(\gamma r)\,d\gamma$$

To satisfy the conditions for σ_z in (1.1), we set

$$-p_\pm(r) = \int_0^\infty \gamma J_0(\gamma r)\,d\gamma \int_0^{r_\pm} \varphi_\pm(t)\cos(\gamma t)\,dt \qquad (1.7)$$

Here $\varphi_\pm(t)$ are new unknown functions, and henceforth we assume them to be continuously differentiable.

Integrating (1.7) by parts and remembering that [3]

$$\int_0^\infty \sin(\gamma t)\,J_0(\gamma r)\,d\gamma = \begin{cases} (t^2 - r^2)^{-1/2}, & r < t \\ 0, & r > t \end{cases}$$

we obtain

$$-p_\pm(r) = \varphi_\pm(r_\pm)(r_\pm^2 - r^2)^{-1/2} - \int_r^{r_\pm} \varphi_\pm'(t)(t^2 - r^2)^{-1/2}dt, \quad r < r_\pm \qquad (1.8)$$

$$p_\pm(r) = 0, \quad r > r_\pm$$

Replacing now the functions $p_\pm(r)$ in the equation for $\bar{p}_\pm(\gamma)$ in (1.4) by their expressions given in (1.7) and remembering that

$$\int_0^\infty J_0(\gamma r)\,\gamma\,d\gamma \int_0^\infty J_0(\gamma\rho)\,\rho f(\rho)\,d\rho = f(r) \quad (0 < r < \infty)$$

we obtain

$$\bar{p}_\pm(\gamma) = -\int_0^{r_\pm} \varphi_\pm(t)\cos(\gamma t)\,dt = -\varphi_\pm(r_\pm)\frac{\sin(\gamma r_\pm)}{\gamma} + \int_0^{r_\pm} \frac{\sin(\gamma t)}{\gamma}\,\varphi_\pm'(t)\,dt$$

The integrand function $K_\pm(\gamma)$ in (1.5) has third order poles at the point $\gamma = 0$, $\gamma J_1(\gamma r)$ is of second order of smallness in γ, and the functions $\bar{p}_\pm(\gamma)$ at this point are different from zero (P is the pressure exerted by the stamps on the layer)

$$p_\pm(0) = \int_0^{r_\pm} \varphi_\pm(r)\,dr = \frac{P}{2\pi} \neq 0$$

$$\left(P = -2\pi \int_0^{r_\pm} \bar{p}_\pm(r)\, r\, dr = 2\pi \int_0^{r_\pm} \varphi_\pm(r)\, r\, dr\right)$$

This means that the integrals

$$\int_0^\infty R_\pm(\gamma)\, \bar{p}_\pm(\gamma)\, \gamma J_1(\gamma r)\, d\gamma$$

diverge, when taken separately. But if we set

$$\bar{p}_\pm(\gamma) = \frac{P}{2\pi} - \Phi_\pm(\gamma), \qquad \Phi_\pm(\gamma) = \int_0^{r_\pm} \varphi_\pm(t)(1 - \cos(\gamma t))\, dt$$

and take into account the fact that the difference $R(\gamma) = R_+(\gamma) - R_-(\gamma)$ is bounded at the point $\gamma = 0$, then we notice that the singularities cancel each other and we obtain

$$I_\pm = \frac{P}{2\pi}\int_0^\infty R(\gamma)\, \gamma J_1(\gamma r)\, d\gamma - \int_0^\infty [R_+(\gamma)\, \Phi_+(\gamma) - R_-(\gamma)\, \Phi_-(\gamma)]\, \gamma J_1(\gamma r)\, d\gamma$$

Having performed these manipulations, we change the order of integration in the integrals of (1.6) to obtain

$$\int_0^r \frac{t\varphi_\pm'(t)\, dt}{r\sqrt{r^2 - t^2}} = \frac{E}{2(1 - \sigma^2)} f_\pm'(r) - \frac{P}{2\pi}\int_0^\infty R(\gamma)\, \gamma J_1(\gamma r)\, d\gamma \mp \int_0^{r_+} \varphi_+'(t)\, Q_\pm(t, r)\, dt \pm \int_0^{r_-} \varphi_-'(t)\, Q_\mp(t, r)\, dt \pm \varphi_+(r_+)\, Q_\pm(r_+, r) \mp \varphi_-(r_-)\, Q_\mp(r_-, r) \tag{1.9}$$

$$Q_\pm(t, r) = \int_0^\infty R_\pm(\gamma)(\gamma t - \sin(\gamma t))\, J_1(\gamma r)\, d\gamma$$

which were derived with help of the relations [3]

$$\int_0^\infty \sin(\gamma t)\, J_1(\gamma r)\, d\gamma = \begin{cases} t\left[r\sqrt{r^2 - t^2}\right]^{-1}, & r > t \\ 0, & r < t \end{cases}$$

2. The operators in the left hand side of the system (1.9) have known inverses. If we assume for the time being that the right hand sides of (1.9) are known and put $t = r\sin\theta$, we obtain the Schlömilch equation

$$\int_0^{\pi/2} F(r\sin\theta)\, d\theta = g(r)$$

the continuous solution of which is

$$F(r) = \frac{2}{\pi}\left[g(0) + r\int_0^{\pi/2} g'(r\sin\theta)\,d\theta\right] \tag{2.1}$$

In fact, the right hand sides of (1.9) are not known, therefore (2.1) generates new integral equations of second kind. These equations are

$$\varphi_{\pm}'(r) = F_{\pm}(r) - \frac{P}{2\pi}\int_0^{\infty} R(\gamma)\gamma\sin(\gamma r)\,d\gamma \mp \int_0^{r_+} \varphi_+'(t)S_{\pm}(t, r)\,dt \pm \tag{2.2}$$

$$\int_0^{r_-}\varphi_-'(t)S_-(t, r)\,dt \pm A_+S_{\pm}(r_+, r) \mp A_-S_{\mp}(r_-, r), \quad A_{\pm} = \varphi_{\pm}(r_{\pm})$$

$$S_{\pm}(t, r) = \frac{2}{\pi}\int_0^{\infty} R_{\pm}(\gamma)(\gamma t - \sin(\gamma t))\sin(\gamma t)\,d\gamma$$

$$F_{\pm}(r) = \frac{E}{\pi(1-\sigma^2)}\int_0^{\pi/2}[r\sin\theta f_{\pm}'(r\sin\theta)]'\,d\theta = \frac{E}{2(1-\sigma^2)}\int_0^{r}\frac{[f_{\pm}'(z)\,z]'}{\sqrt{r^2-z^2}}\,dz$$

It can be shown that the kernels $S_{\pm}(t, r)$ are continuous in the square $[0, r_+] \times [0, r_-]$, therefore the system is Fredholmian.

In what follows, it is expedient to introduce the quantities $\varepsilon_{\pm} = r_{\pm}/d$, $u = \gamma d$, $x = r/d$ and put

$$\varphi_{\pm}'(r) = \frac{P}{2\pi d^2}\psi_{\pm}(x)$$

It can be seen that in the limit as $d \to \infty$ (the case of a half-space), the equations (2.2) become

$$\varphi_{\pm}'(r) = F_{\pm}(r)$$

and we obtain the following solution of the problem for a half-space (see (1.8)):

$$\sigma_z = \frac{\varphi_-(r_-)}{\sqrt{r_-^2 - r^2}} - \int_r^{r_-}\frac{F_-(t)}{\sqrt{t^2-r^2}}\,dt = \frac{A_-}{\sqrt{r_-^2-r^2}} -$$

$$\frac{E}{\pi(1-\sigma^2)}\int_0^{r}\frac{[f_-'(t)\,t]'}{\sqrt{r^2-t^2}}\,dt$$

In particular, for a flat stamp we have

$$f_-'(r) = 0, \quad \sigma_z = \frac{P}{2\pi r_-\sqrt{r_-^2-r^2}}$$

therefore

$$\lim_{d\to\infty} A_\pm = \frac{P}{2\pi r_\pm}$$

Hence we set

$$A_\pm = \frac{P}{2\pi r_\pm} B_\pm$$

In the new notation the system (2.2) becomes (restricting ourselves to the case of flat stamps):

$$\psi_\pm(x) = \int_0^\infty R^\circ(u)\, u \sin(ux)\, du \mp \int_0^{\varepsilon_+} \psi_+(t)\, T_\pm(t, x)\, dt \pm \tag{2.3}$$

$$\int_0^{\varepsilon_-} \psi_-(t)\, T_\mp(t, x)\, dt \pm \frac{B_+}{\varepsilon_+} T_\pm(\varepsilon_+, x) \mp \frac{B_-}{\varepsilon_-} T_\mp(\varepsilon_-, x)$$

$$R^\circ(u) = \frac{2u + 1 - e^{-2u}}{\operatorname{sh} 2u + 2u}$$

$$T_\pm(t, x) = \frac{2}{\pi}\int_0^\infty U_\pm(2u)\,(ut - \sin(ut))\sin(ux)\, du$$

$$U_+(z) = \frac{e^{-z}\operatorname{sh} z + z^2 + z}{(\operatorname{sh} z)^2 - z^2}$$

$$U_-(z) = \frac{z\operatorname{ch} z + \operatorname{sh} z}{(\operatorname{sh} z)^2 - z^2}$$

3. The system (2.3) can be solved using various approximate methods. We shall use the simplest method of small parameter. Expanding $\sin(ux)$ into a power series, we obtain

$$T_\pm(t, x) = a_\pm(t)\, x + b_\pm(t)\, x^3 + \ldots \tag{3.1}$$

Taking into account the fact that

$$U_+(z) \sim 4z^2 e^{-2z}, \quad U_-(z) \sim 2ze^{-z}, \quad z \to +\infty$$

we can show that the series (3.1) converge when $\varepsilon_0 \equiv \max(\varepsilon_-, \varepsilon_+) < 2$, and the smaller value of ε_0 the better convergence. For an iterative process to converge, it is sufficient that

$$\alpha \equiv \| T_+ \| + \| T_- \| < 1$$

$$(\| T_\pm \| = \max_\Delta |T_\pm(x, t)|, \ \Delta = [0 \leqslant x \leqslant \varepsilon_0] \times [0 \leqslant t \leqslant \varepsilon_0])$$

Separating the interval $[0, \infty)$ into two subintervals $[0, a]$ and (a, ∞) and approximating the functions $U_\pm(z)$ in a suitable manner on each of these subintervals, we can show that $\alpha \leqslant 0.88\,\varepsilon_0^3$. Therefore the iterative process will certainly converge if $\varepsilon_0 \leqslant 1$.

Under these conditions the solution of (2.3) can be written in the form of series in powers of ε_+ and ε_-. To obtain approximate solutions with terms not greater than ε_0^5, we must compute the integrals

$$f_{\pm}^{(j)} \equiv \int_0^{\infty} R_{\pm}^{\circ}(u)\, u^{2j} du \quad (j = 1, 2) \tag{3.2}$$

This was done by means of a computer, with the function $\overset{\circ}{R}_{\pm}(u)$ approximated each time so that the error in computing the integrals (3.2) did not exceed 0.001. This gave the following approximation formulas for the normal stresses at the zone of contact:

$$\sigma_z \big|_{z=\pm d} = \frac{P}{2\pi r_{\pm} \sqrt{r_{\pm}^2 - r^2}} \left(1 + 0.0866\varepsilon_{\pm}^3 - 0.1732\varepsilon_{\pm} x^2 + 0.0323\varepsilon_{\pm}^5 - \ldots\right)$$

REFERENCES

1. Ufliand Ia. S. Integral Transforms in the Problems of the Theory of Elasticity. Leningrad, Nauka, 1967.

2. Vorovich, I. I., Aleksandrov, V. M. and Babeshko, V. A., Nonclassical Mixed Problems of the Theory of Elasticity. Moscow, "Nauka", 1974.

3. Gradshtein, I. S. and Ryzhik, I. I., Tables of Integrals, Sums, Series and Products. Moscow, Fizmatgiz, 1963. (see also English translation, Pergamon Press, Book No. 09832, 1963).

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